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A Comprehensive Review on Burger's Equation and Lie Group Theory

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Abstract—The present article reviews the various forms of the Burger's equation and their well-known results/solutions obtained from Lie group analysis with their recent contributions in non-linear partial differential equations (PDEs). This study presents a conventional assessment of the preceding work but also produce all helpful existing information that can help for the solution of Burger's equation. Burger's equation describes the propagations of non-linear waves in fluid mechanics and it is the simplest non-linear model for soil water turbulent flow pattern, hydrodynamics turbulence (in shock waves). Also, Lie symmetry of PDEs is one parameter group of transformations, which reduces the number of independent variables and leaves the PDEs invariant. In addition, a brief discussion of previous works on Burger's equation and Lie symmetry analysis has also been presented in the article.

Keywords: *Burger's equation, similarity transformation methods, invariant solutions, Lie group theory.*

1. INTRODUCTION:

Bateman[10] first introduced Burger's equation which was later in 1948 analyzed by Burger. The equation is used as a model in various fields, for example, gas dynamics modelling, shallow water-waves investigation, traffic flow, acoustics, continuous stochastic processes, dispersive water, heat conduction. The equation can also be considered as the Navier-Stokes equation in simplified form due to the occurrence of the viscous term and convection term. In fluid dynamics, initially, it was also used for the turbulent motion of fluids in statistical theory context. Harry Bateman [10], introduced a partial differential equation (PDE) with the preconditions given by:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\nu\frac{\partial^2 u}{\partial x^2},\ 0<x<l,\ 0<t<\lambda \tag{1}$$

$$u(x,0)=\phi(x),\ 0<x<l \tag{2}$$

$$u(0,t)=\psi_1(t)\ ,u(l,t)=\psi_2(t)\ ,\qquad 0<t<\lambda \tag{3}$$

Where u is velocity, x is spatial coordinate, t is time and ν is kinematic viscosity, respectively.

ϕ, ψ_1, ψ_2 are functions of variables (based on some preconditions) for the problem to be solved. For small-scale viscosity as a result of moderate convergence of solutions, numerical approach to discover the solutions of Burger's equation was found impractical and for the extension of the corresponding initial and boundary conditions, the exact solutions of Burger's equation have been obtained instead of its non-linearity. Some exact solutions were obtained by A. Ouhadan et al. [11] forming a new class of Lie point symmetries through its maximal sub-algebras like rational, exponential and periodic. However, the required solutions of the forced Burger's equation [12] is built for stationary and transient external forces by using modified Hopf-Cole transformation. These researches motivate us to discover exact solutions of Burger's equation by using similarity transformations methods(STM).

1.1 PHYSICAL CONTEXT:

Burger's equation is the most celebrated equation as it involves both the effects of non-linear propagation and effects of diffusion. The presence of together $u\frac{\partial u}{\partial t}$ (which is the non-linear convective term) and $\nu\frac{\partial^2 u}{\partial x^2}$ (which is the diffusive term) adds an extra feature to the Burger's equation. As ν tends to zero Eqn (1) becomes in viscid Burger's equation used as a prototypal for non-linear wave propagation, and when u tends to zero, Eqn (1) becomes the heat equation.

Burger's equation, very fundamental PDE, produce an exceptional example because:

1. This is used in the calculation of the boundary layer for viscous fluid flow.
2. It forms a standard test problem for PDE solvers.
3. It is used for small-scale kinematic viscosity u, it is the situation with artificial diffusion.
4. It governs the growth of molecular interfaces, non-linear wave propagation, sedimentation of poly-disperse suspensions and colloids, longitudinal elastic waves in isotropic solids, traffic flow, cosmology, an aspect of turbulence, gas dynamics and shock wave theory.

1.2. FORMATION OF LIE GROUPS OF DIFFERENTIAL EQUATIONS :

For a given differential equation ψ having n independent variables as x = $(x_1,..., x_n) \in \Re^n$, and m dependent variables u =$(u_1,\ldots,u_m) \in \Re^m$. In consideration of Lie group of point transformation that associates with the given differential equation ψ. Transformations take place:

$$\mathbf{x}^* = \mathbf{X}(\mathbf{x}, \mathbf{u}; a), \mathbf{u}^* = \mathbf{U}(\mathbf{x}, \mathbf{u}; a) \text{ acting on } \Re^{n+m} \text{ of the variables } (\mathbf{x}, \mathbf{u}). \tag{4}$$

$$\text{Let } \mathbf{u} = \theta(\mathbf{x}) \equiv (\theta_1(x), \theta_2(x), \ldots, \theta_m(x)) \tag{5}$$

The group transformation maps any result of ψ into a different result of ψ. The group transformation leaves ψ invariant, ψ is the same in both the terms of the transformed variables (x^*, u^*) and in terms of the variable (**x, u**).

Let $u^{(1)}$ denote the set of all *m*.n first order partial derivatives of **u** with respect to **x** as:

$$u^{(1)} \equiv \left(\frac{\partial u^1}{\partial x^1}, \ldots, \frac{\partial u^1}{\partial x^n}, \ldots, \frac{\partial u^m}{\partial x^1}, \ldots, \frac{\partial u^m}{\partial x^n}\right) \tag{6}$$

and, let $u^{(k)}$ represents the set of all $k^{(th)}$-orderpartial derivatives of **u** with respect to **x**.

The one-parameter Lie group of transformations (4) acts on the space (**x, u**), and the extended group acts on the space (x, u, $u^{(1)}$) and the jet space (x, u, $u^{(1)}$, $u^{(k)}$) as transformations of derivatives of dependent variables lead to natural extensions of one-parameter Lie group of transformations. As all the information about a Lie group of transformations is contained in its infinitesimal generator, we need to compute it's extensions:

• The first-order extension

$$\chi^{(1)} = \chi + \sum_{J=1}^{m} \sum_{i=1}^{n} \xi^{J}_{[i]}(x, u, u^{(1)}) \frac{\partial}{\partial u^{J}_{i}}, \quad where \quad u^{J}_{i} = \frac{\partial u^{J}}{\partial x_i} \tag{7}$$

with

$$\chi^{J}_{[i]}\left(x, u, u^{(1)}\right) = \frac{D\chi^{J}}{Dx_1} - \frac{D\varphi_j}{Dx_i}\frac{\partial u^{J}}{\partial x_j} \tag{8}$$

The general k^{th}-order extensions iteratively defined by :

$$\chi^{k)} = \chi^{(k-1)} + \sum_{J=1}^{m} \sum_{i_1=1}^{n} \cdots \sum_{i_k=1}^{n} \xi^{J}_{[i_1 \ldots i_k]} \frac{\partial}{\partial u^{J}_{i_1 \ldots i_k}}, \tag{9}$$

$$where \quad u^{J}_{[i_1 \ldots i_k]} = \frac{\partial^k u^{J}}{\partial x_{i_1} \ldots \partial x_{i_k}} \tag{10}$$

and $\xi^{J}_{[i_1 \ldots i_k]}$, defined iteratively by the relation

$$\xi^{J}_{[i_1 \dots i_k]} = \frac{D\,\xi^{J}_{[i_1 \dots i_{k-1}]}}{Dx_{i_k}} - u^{J}_{i_1 \dots i_{k-1} j}\,\frac{D\varphi_j}{Dx_{i_k}} \tag{11}$$

In (8) and (10) the Lie derivative

$$\frac{D}{Dx_i} = \frac{D}{\partial x_i} + \frac{\partial u^J}{\partial x_i}\frac{\partial}{\partial u^J} + \frac{\partial^2 u^J}{\partial x_i \partial x_j}\frac{\partial}{\partial u^J_j} + \dots \tag{12}$$

has been introduced, and over repeated indices, Einstein convention of summation is used. One-parameter Lie groups of transformations leaves differential equations invariant.

Let

$$\Delta(x,u,u^{(1)},\dots,u^{(k)}) = 0, \qquad \Delta_\nu(x,u,u^{(1)},\dots,u^{(k)}) \equiv u^{J_\nu}_{i_1 \dots i_{k_\nu}} - \Theta(x,u,u^{(1)},\dots,u^{(k)}) = 0 \tag{13}$$

The equations (12) can be considered as characterizing a sub-manifold in the k^{th}- order jet space, the latter having dimension equal to

$$n + m\sum_{h=0}^{k}\binom{n+h-1}{n-1} = n + m\binom{n+k}{k} \tag{14}$$

It is said that one parameter Lie group of transformations (4) leaves the system (12) invariant if and only if its k^{th} prolongation leaves invariant the sub-manifold of the jet space defined by (12).

2. ANALYSIS AND SURVEY:

In the field of non-linear (PDE) currently,transformation techniques are very useful. This is mostly used in point transformation and in the space of both the dependent and independent variables of PDE. The PDEs remains invariant and forms continuous Lie group under point transformations. Applying similarity reductions the solutions of the PDE are reduced into ordinary differential equation (ODE).

The non-linear diffusion-convection equations of the type [13]

$$u_t = (A(u)u_x)_x + C(u)u_x \tag{15}$$

named as Richards equations, has many applications in the field of mathematical physics and is useful in the learning of porous media.

Now if we consider (2+l) dimensional non-linear diffusion-convection equations it takes the form:

$$u_t = (A(u)u_x)_x + (B(u)u_y)_y + C(u)u_x \tag{16}$$

and the (3+1)-dimensional takes the form:

$$u_t = (A(u)u_x)_x + (B(u)u_y)_y + (D(u)u_z)_z + C(u)u_x \tag{17}$$

Where A(u), B(u), C(u) and D(u) are random smooth functions, $B(u) \neq 0$, $D(u) \neq 0$ and $A^2(u) + C^2(u) \neq 0$ (otherwise lower dimension class will formed) and $A_u^2 + B_u^2 + C_u^2 + D_u^2 \neq 0$ (equation will be non-linear).

Boisvert et al. [16], Nucci [17] and Ames [15] constructed a kind of transformation that permit the transformation of the time-dependent equation into it's corresponding time- independent equation with the presumptions of particular Lie-group with random functions and from which steady equations results generates an infinite number time-dependent solutions.

Mazzia [14] applied transverse scheme where results were defined along the time axis with the combination of boundary value problems successfully converted Burger's equation into ODEs system.

Kumar [3] constructed the exact solutions in terms of Bessel functions of 1-D Burger's equation by using STM. The author has shown the nature of solutions graphically and made some discussion.

The authors have considered the 1-D Burger's equation as:

$$u_t + uu_x = \nu u_{xx}, \quad \nu > 0 \tag{18}$$

Here u(x, t) is velocity and v is kinematic viscosity of the fluid. Eq.(17) is a quasi-linear parabolic PDE. Kumar et al. studied Eq.(17) by using a similarity transformation method and obtained various exact solutions. The solutions represent the solution behaviour.

Doyle [19] applied similarity technique to solve the generalized Burgers equation where the author forms three parameter symmetry groups arising from the three individual cases of the Burger's equation. Reduced form to ODE and the formed Lie algebras of the groups are given.

Djordjevic [18] examined similarity solutions using Lie-group transformation. Solutions of a nonlinear fractional diffusion equation and fractional Burgers/Korteweg-deVries equation in one spatial variable have been examined. For the nonlinear heat conduction equation author discover two different types of similarity transformations. One of them has the form of a wave propagating in one direction with specific speed. A relation between speed and amplitude is also made.

Wazwaz [5] aimed to study the (2+l)-dimensional integrable Burger's equation. It has two forms. The first one is the (2+l)-dimensional Painlev'e integrable Burger's equation that reads

$$u_t = uu_y + \lambda v u_x + \mu u_{yy} + \lambda\mu u_{xx}, \quad u_x = v_y \tag{19}$$

where λ and μ are nonzero constants. The second one :

$$u_t = u_{xx} + u_{yy} + 2uu_x + 2vu_y, \;\; v_t = v_{xx} + v_{yy} + 2uv_x + 2vv_y \tag{20}$$

Multiple kink results and multiple singular kink results are formally examined for this equation. The procedure is mainly based on Hirota's bilinear method.

Pandiaraja [4] employed symmetry classification of the 2-D Burger's equation with variable coefficient. He found symmetry algebra and sub algebras, up to conjugacy, is made. Similarity reductions are made in every class.

Some important form of the (2+1) dimensional Burger's equations are given below:

The (2+l)-dimensional Burger's equation:

$$v_t = v_{xx} + 2v_x \partial_y^{-1} v_x \tag{21}$$

The (2+l)-dimensional high-order Burger's equation:

$$v_t = 4v_{xxx} + 12v_{xx}\partial_y^{-1}v_x + 12v_x\partial_y^{-1}v_{xx} + 12v_x\left(\partial_y^{-1}v_x\right)^2 \tag{22}$$

The derived generalized Burger's equation was validated to be Painlev'e integrable given in the following form

$$u_t = \alpha uu_x + \beta u_{xx} + \gamma vv_x + \frac{\beta\gamma}{\alpha} v_{xy}\,,\; u_y = v_x \tag{23}$$

where α, β and, γ are non-zero constants.

Christou [6] made the list of similarity reduction obtained from the Lie symmetries produced by the equation $u_t = u_{xx} + u_{yy} + u_{zz} + uu_x$. The author employs 2-D and 3-D sub-algebras of the Lie symmetry algebra, additionally the 1-D subalgebras that turn up in the literature and solved theoretically, hence resulted in closed-form solutions for the original equation. At last, the (3+l)-dimensional Burger's equation takes the form :

$$u_t = \alpha uu_y + \beta vu_x + \gamma wu_x + u_{xx} + u_{yy} + u_{zz}, u_x = v_y, \quad u_x = w_y \tag{24}$$

where α , β and γ are nonzero-constants. For w = 0, and if u is independent of z, Eq. (25) will be reduced into the (2 + l)-dimensional Burger's equation.

Sophus Lie [20] introduced the concept of symmetry of a differential equation. Although Lie has obtained various results due to broader applications of symmetry reduction at the end of the 19th century. The author finds the nonlinear temperature adaptation over the lake depth neglecting the effect of external heat sources. The steady and the unsteady laminar boundary-layer flow of a non-isothermal vertical circular cylinder have been studied by Abd-el-Malek [20] and Badran in 1990 and 1991.

3. CONCLUSIONS:

A brief review of the various forms of Burger's equation and the similarity transformation method is presented in this paper. All the forms of Burger's equation reviewed in this manuscript are reduced into system of ODEs by using similarity transformation which provides exact solutions. Also, some physical applications of Burger's equation such as boundary layer problem, the problem with artificial diffusion, traffic flow are cited. Thus, this paper may be helpful to provide the research gap in Burger's equation for future research.

REFERENCES:

[1]. G.W. Bluman, J.D. Cole, Similarity Methods for Differential Equations, Springer–Verlag, New York, 1974.
[2]. P.J. Olver, Applications of Lie Groups to Differential Equations, Springer–Verlag, New York, 1993.
[3]. M. Kumar, R. Kumar, Some More Solutions of Burger's Equation, Phys.: Conference Series 574 (2015) 012038, IOP Publishing, ISSN: 1742-6588, DOI:10.1088/1742-6596/574/1/012038.
[4]. D. Pandiaraja, B. MayilVaganan, Symmetry Analysis of 2+1 dimensional Burger's equation with variable damping, Article arXiv(2010).
[5]. Abdul-Majid Wazwaz, A study on a (2 + 1)-dimensional and a (3 + 1)-dimensional generalized Burger's equation, Applied Mathematics Letters, 31 (2014) 41–45.
[6]. M.A. Christou, N.M. Ivanova, C. Sophocleous, Applied Mathematics and Computation 210 (2009) 87–99.
[7]. H. Ke-Zhu, Wu B-in, C. Xiang-Feng, Painlev´e analysis and some solutions of the (2 + 1) -dimensional generalized Burger's equations, Commun. Theor. Phys. 39 (2003) 393–394.
[8] A.M. Wazwaz, Multiple soliton solutions and multiple singular soliton solutions for the (3+1)-dimensional Burger's equations, Appl. Math. Comput. 204 (2008) 942–948.
[9] Christodoulos Sophocleous, Chaos, Solitons and Fractals 20 (2004) 1047–1057, Transformation properties of a variable-coefficient Burger's equation.
[10] Mayur P Bonkile, Ashish Awasthi, C Lakshmi, VijithaMukundan , V S Aswin , A systematic literature review of Burger's equation with recent advances, Pramana–J.Phys.(2018)90:69.
[11] Ouhadan A, Mekkaoui T, Kinani E H, El 2011 Int. J Nonlinear Sci. 11 51.
[12] Petrovskii S V, 1999 Tech. Phys. 44 878.
[13] E. Demetriou, N.M. Ivanova, C. Sophocleous, Group analysis of (2+1) and (3+1)-dimensional diffusion–convection equations, J. Math. Anal. Appl. 348 (2008) 55–65.
[14] A. Mazzia, F. Mazzia, Higher-order transverse schemes for the numerical solution of partial differential equations, J. Comput. Appl. Math. 82 (1997) 299–311.
[15] W.F. Ames, M.C. Nucci, Analysis of fluid equations by group method, J. Engrg. Math. 20 (1985) 181–187.
[16] R.E. Boisvert, W.F. Ames, U.N. Srivastava, Group properties and new solutions of Navier–Stokes equations, J. Eng. Math. 17 (1983) 203–221.
[17] M.C. Nucci, Group analysis of MHD equations, Atti Semi Mat. Fis. Univ. Modena 33 (1984) 21–34.
[18] Vladan D. Djordjevic, Teodor M. Atanackovic, Similarity solutions to nonlinear heat conduction and Burgers/Korteweg–deVries fractional equations, Journal of Computational and Applied Mathematics 222 (2008) 701–714.
[19] J. Doyle, M. J. Englefield, Similarity Solutions of a Generalized Burgers Equation, IMA Journal of Applied Mathematics (1990) 44, 145-153.
[20] Abd-el-Malek, M.B., Group theoretic methods applied to Burger's equation, Journal of Computational and Applied Mathematics 20000301.
[21] G.W. Bluman, S. Kumei, Symmetries and differential equations, Springer, New York, 1989.
[22] L.V. Ovsiannikov, Group analysis of differential equations, Academic Press, New York, 1982.
[23] Francesco Oliveri, Lie Symmetries of Differential Equations: Classical Results and Recent Contributions Journal Symmetry 2010, 2, 658-706.
[24] J.P. Ying, S.Y. Lou, Multilinear variable separation approach in (3+1)-dimensions: the Burger's equation,Chin. Phys. Lett. 20 (2003) 1448–1451. [25] S.S. Peng, P.Z. Liang, Z. Jun, New exact solution to (3 + 1)-dimensional Burgers equation, Commun. Theor. Phys. 42 (2004) 49–50.